An Abstraction-Refinement Methodology for Reasoning about Network Games

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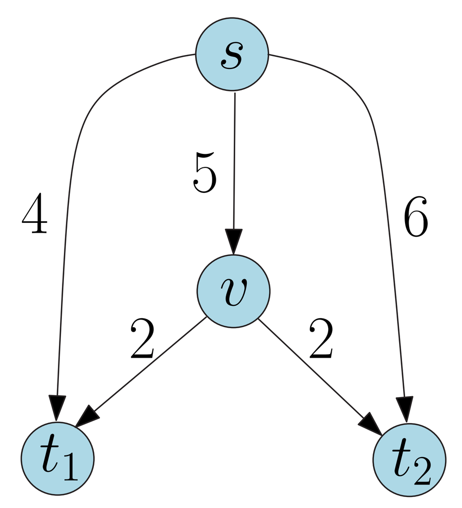
Network games [3]

- A network game (NG) is played on a **weighted directed graph**.
- **Multiple players**; each player has to find a **path from a source to a target**.
- A **strategy** is a path of a player from her source to destination.
- In a **cost-sharing game** (CS-NG), the players share the cost of an edge.
- A **profile** is a tuple of strategies, one for each player.
- In a profile, a **player pays for the edges she uses**.
- The **cost of a profile** is the sum of the costs of all the players.
- A **social optimum SO** is a **cheapest profile**.
- An **NE** is a **stable profile** from which no player can make a beneficial move unilaterally.



1	2	1 pays	2 pays	Total
Outer	Outer	4	6	10
Outer	Middle	4	7	11
Middle	Outer	7	6	13
Middle	Middle	$5/2+2$	$5/2+2$	9

SO: ⟨**Middle, Middle**⟩, **NE**: ⟨**Outer, Outer**⟩

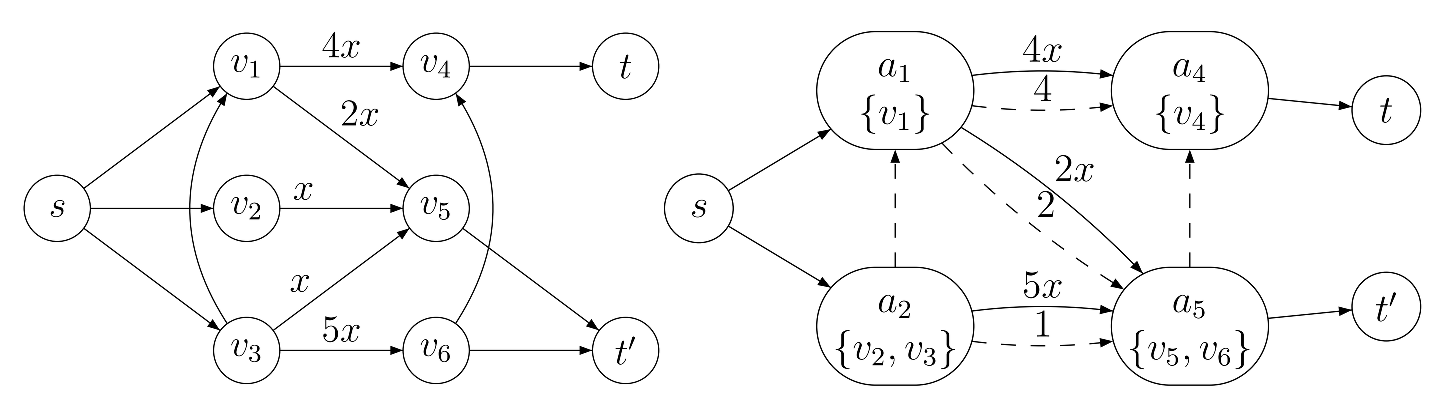
- **Congestion** cost function: e.g. $\mathbf{f(x) = ax + b}$.

Under- and Over-approximations $\mathcal{N}^{\downarrow}[\alpha]$ and $\mathcal{N}^{\uparrow}[\alpha]$of an NG $\mathcal{N}$

- In **$\mathcal{N}^{\downarrow}$, each player has fewer strategies and pays at least as much** as in $\mathcal{N}$. In **$\mathcal{N}^{\uparrow}$, each player has more strategies and pays not more** than in $\mathcal{N}$.
- **Transition Relations**: $\mathbf{E^{\downarrow}(a, a')}$ iff for every concrete vertex $\mathbf{v} \in \mathbf{a}$, there is a concrete vertex $\mathbf{v'} \in \mathbf{a'}$ such that $\mathbf{E(v, v')}$. $\mathbf{E^{\uparrow}(a, a')}$ iff there exist concrete vertices $\mathbf{v} \in \mathbf{a}$ and $\mathbf{v'} \in \mathbf{a'}$ such that $\mathbf{E(v, v')}$.
- **Cost functions**:

	$\mathcal{N}^{\downarrow}$	$\mathcal{N}^{\uparrow}$
Transitions	Must	May
Cost	Max	Min
Effect of load in CS-NG	1	Sum
Effect of load in CON-NG	Sum	1

An Example



A **CON-NG** $\mathcal{N}$ (left) and its **approximations $\mathcal{N}^{\downarrow}$ and $\mathcal{N}^{\uparrow}$** (right). Edges in $\mathbf{E}^{\downarrow}$ are solid. Edges in $\mathbf{E}^{\uparrow} \setminus \mathbf{E}^{\downarrow}$ are dashed. Edges with no specified cost have cost **0**.

Objective

Find an SO and an NE of an NG by reasoning about its under- and over-approximations.
Inputs: An NG $\mathcal{N}$, and an abstraction function $\alpha : \mathbf{V} \rightarrow \mathbf{A}$ that abstracts the set $\mathbf{V}$ of vertices to a **smaller set A** of abstract vertices.

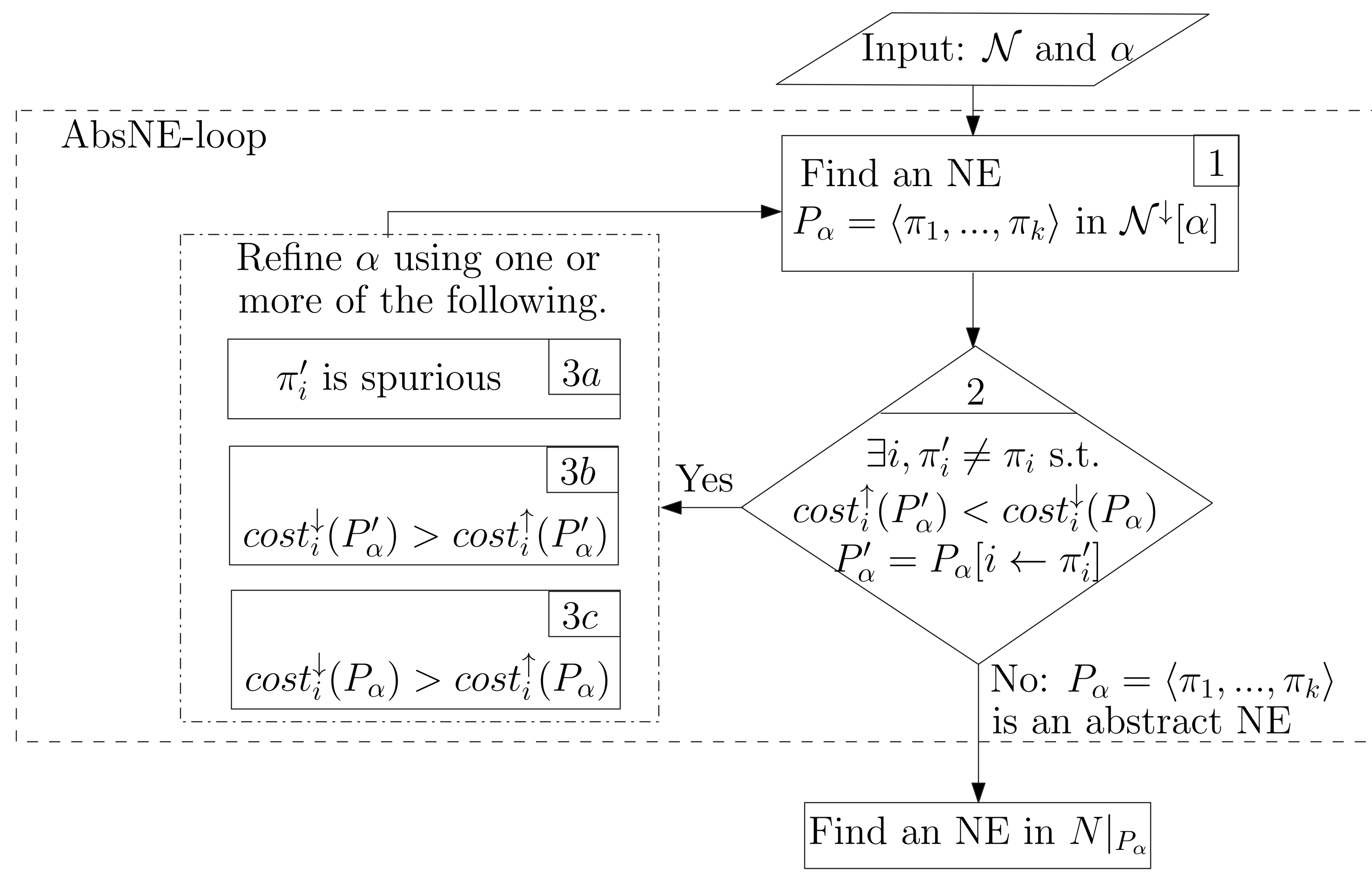
- **Theorem: There exists an NE** in every NG [3].
- **Theorem: Complexity** of finding an NE is **PLS-complete** [2].
- **Counterexample guided abstraction refinement (CEGAR)** has been successfully used in **verification** to reason about systems with **large state space [1]**.

Find an SO and an NE in $\mathcal{N}$ using $\mathcal{N}^{\downarrow}$ and $\mathcal{N}^{\uparrow}$

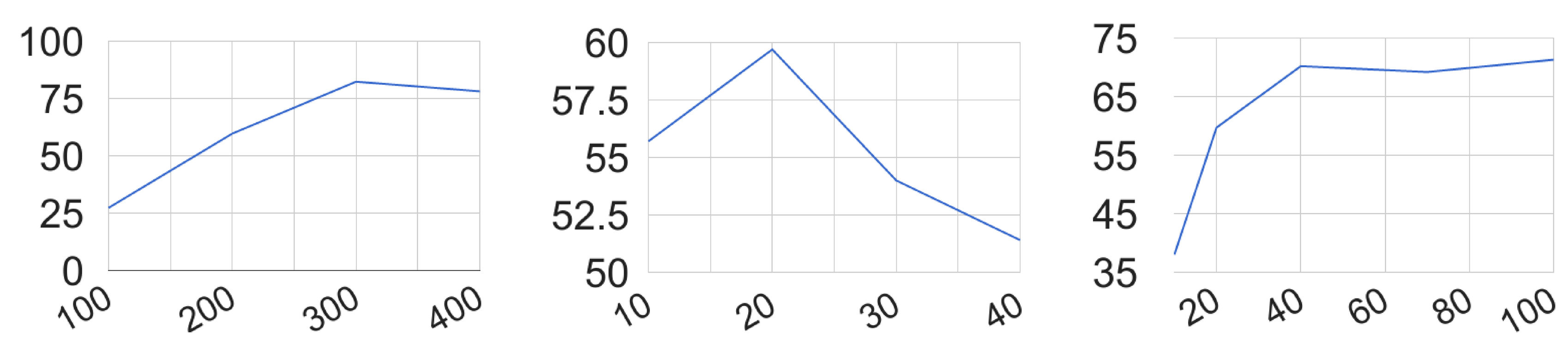
- **Theorem**: If $\alpha_2 \preceq \alpha_1$, then $\mathbf{SO(\mathcal{N}^{\downarrow}[\alpha_2]) \leq SO(\mathcal{N}^{\downarrow}[\alpha_1])}$ and $\mathbf{SO(\mathcal{N}^{\uparrow}[\alpha_1]) \leq SO(\mathcal{N}^{\uparrow}[\alpha_2])}$, i.e. **successive refinements reduces the gap between the upper and the lower bounds** of an SO in $\mathcal{N}$.
- **Abstract NE**: An NE in $\mathcal{N}^{\downarrow}$ such that no player has beneficial deviation even in $\mathcal{N}^{\uparrow}$.
- **Theorem**: Consider an **abstract NE P** in $\mathcal{N}^{\downarrow}[\alpha]$. There exists a **profile in α^{-1}(P)** that is a **concrete NE** in $\mathcal{N}$.

An Abstraction-Refinement Framework to Find an NE

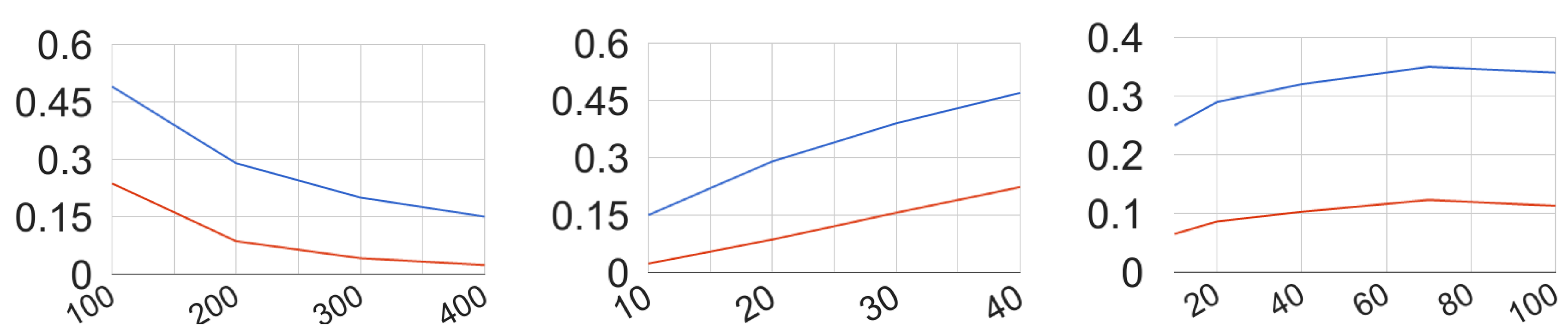
- Find an **abstract-NE using an abstraction-refinement framework**.



Experimental Results



The **number of iterations to find an abstract-NE** (**y**-axis) as $|\mathbf{V}|$, $\mathbf{k}$, and $|\mathbf{W}|$ increase (**x**-axis); $|\mathbf{V}|$: number of vertices, $\mathbf{k}$: number of players, and $|\mathbf{W}|$: range on weights on the edges.



The **ratio between the size (vertices and edges)** of the concrete and truncated networks, namely, $\mathcal{N}_{|\mathbf{P}_\alpha}$ (**y**-axis) as $|\mathbf{V}|$, $\mathbf{k}$, and $|\mathbf{W}|$ increase (**x**-axis).
The **blue lines** indicate the ratios between the **vertices** while the **red lines** indicate the ratios between the **edges**.

References: Bibliography

E. M. Clarke, O. Grumberg, S. Jha, Y. Lu, and H. Veith.
Counterexample-guided abstraction refinement for symbolic model checking.
Journal of the ACM, 50(5):752–794, 2003.

A. Fabrikant, C. Papadimitriou, and K. Talwar.
The complexity of pure nash equilibria.
In *Proc. 36th ACM Symp. on Theory of Computing*, pages 604–612, 2004.

R. W. Rosenthal.
A class of games possessing pure-strategy Nash equilibria.
International Journal of Game Theory, 2:65–67, 1973.